Support Vector Machines Classification with Robust Chance Constraints

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Outline

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   - SVM with Robust Chance Constraints
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   - RCC-SVM into SOCP Models
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4. Conclusions

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Hard Margin SVM

$$\begin{aligned} \min_{\mathbf{w},b} \quad & \frac{1}{2}\|\mathbf{w}\|_2^2 \\ \text{s.t.} \quad & y_i(\mathbf{w}^\top \mathbf{x}_i + b) \geq 1, \quad i = 1, \ldots, m \end{aligned}$$

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- When $0 < \xi_i < 1$, the data is within margine but correctly classified; when $\xi_i > 1$, the data is misclassified.

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Soft Margin SVM

$$\min_{\mathbf{w},b,\xi_i} \ \frac{1}{2}\|\mathbf{w}\|_2^2 + C\sum_{i=1}^{m}\xi_i$$

$$\text{s.t.} \ \ y_i(\mathbf{w}^\top \mathbf{x}_i + b) \geq 1 - \xi_i, \quad \xi_i \geq 0, \quad i = 1, \ldots, m$$

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$$\min_{\mathbf{w},b,\xi_i} \; \frac{1}{2}\|\mathbf{w}\|_2^2 + C\sum_{i=1}^{m}\xi_i$$

$$\text{s.t.} \;\; \mathbb{P}\left\{y_i(\mathbf{w}^\top\tilde{\mathbf{x}}_i + b) \le 1 - \xi_i\right\} \le \varepsilon, \quad \xi_i \ge 0, \quad i = 1, \dots, m$$

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- Combine the first and second moments $\boldsymbol{\Sigma}_i, \boldsymbol{\mu}_i$ into one matrix Ω_i:

$$\Omega_i = \begin{bmatrix} \boldsymbol{\Sigma}_i + \boldsymbol{\mu}_i \boldsymbol{\mu}_i^\top & \boldsymbol{\mu}_i \\ \boldsymbol{\mu}_i^\top & 1 \end{bmatrix}$$

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- Let $\mathcal{P}$ be the set of all probability distributions that have the same first and second moments.

Theorem

RCC-SVM is equivalent to the following SDP formulation:

$$\min_{\mathbf{w},b,\xi_i,\mathbf{N}_i,\alpha_i} \quad \frac{1}{2}\|\mathbf{w}\|_2^2 + C\sum_{i=1}^{m}\xi_i$$

$$s.t. \quad \alpha_i - \frac{1}{\varepsilon} Trace(\Omega_i \mathbf{N}_i) \geq 0, \quad \xi_i \geq 0$$

$$\mathbf{N}_i \succeq 0, \quad \mathbf{N}_i + \begin{bmatrix} 0 & \frac{1}{2}y_i\mathbf{w} \\ \frac{1}{2}y_i\mathbf{w}^\top & y_i b + \xi_i - 1 - \alpha_i \end{bmatrix} \succeq 0$$

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Multivariate Chebyshev Inequality

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- The multivariate Chebyshev inequality states that for an arbitrary closed convex set S, the supremum of the probability that $\tilde{\mathbf{x}}$ takes a value in S is

$$\sup_{\tilde{\mathbf{x}} \sim (\boldsymbol{\mu}, \boldsymbol{\Sigma})} \mathbb{P}\{\tilde{\mathbf{x}} \in S\} = \frac{1}{1 + d^2}$$

$$d^2 = \inf_{\mathbf{x} \in S} (\mathbf{x} - \boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1} (\mathbf{x} - \boldsymbol{\mu})$$

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- Using multivariate Chebyshev inequality, the SOCP reformulation of RCC-SVM is:

$$\min_{\mathbf{w},b,\xi_i} \quad \frac{1}{2}\|\mathbf{w}\|_2^2 + C\sum_{i=1}^{m} \xi_i$$

$$\text{s.t.} \quad y_i(\mathbf{w}^\top \boldsymbol{\mu}_i + b) \geq 1 - \xi_i + \sqrt{\frac{1-\varepsilon}{\varepsilon}}||\boldsymbol{\Sigma}_i^{\frac{1}{2}}\mathbf{w}||_2$$

$$\xi_i \geq 0, \quad i = 1, \ldots, m$$

Geometric Interpretation of the SOCP Model

- For each point $\mathbf{x}_i$, it is no longer a single point, but an ellipsoid centered at $\boldsymbol{\mu}_i$, and shaped with the covariance matrix $\boldsymbol{\Sigma}_i$:

$$\mathscr{E}(\boldsymbol{\mu}_i, \boldsymbol{\Sigma}_i) = \left\{\mathbf{x} = \boldsymbol{\mu}_i + \sqrt{\frac{1-\varepsilon}{\varepsilon}}\boldsymbol{\Sigma}_i^{\frac{1}{2}}\mathbf{a} \ : \ ||\mathbf{a}||_2 \le 1\right\}$$

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- The SOCP constraint is satisfied if and only if

$$y_i(\mathbf{w}^\top \mathbf{x}_i + b) \geq 1 - \xi_i, \quad \forall \mathbf{x}_i \in \mathscr{E}\left(\boldsymbol{\mu}_i, \boldsymbol{\Sigma}_i\right)$$

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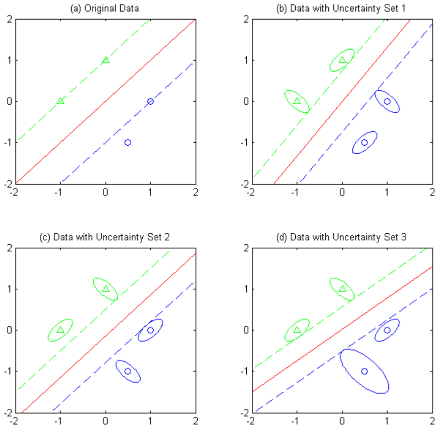
$$y_i(\mathbf{w}^\top \mathbf{x}_i + b) \geq 1 - \xi_i, \quad \forall \mathbf{x}_i \in \mathscr{E}(\boldsymbol{\mu}_i, \boldsymbol{\Sigma}_i)$$

- This transforms the RCC-SVM into a robust optimization problem over the uncertainty set $\mathscr{E}(\boldsymbol{\mu}_i, \boldsymbol{\Sigma}_i)$ for each uncertain training data point.

Geometric Interpretation of the SOCP Model



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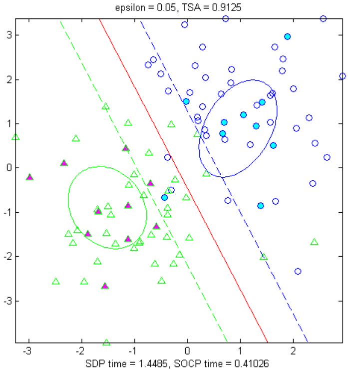
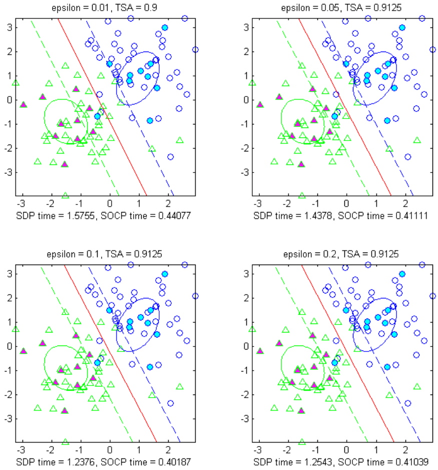
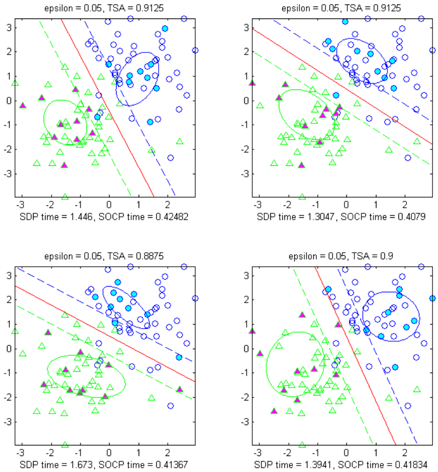
$+1$ class: 2-d normal distribution with $\boldsymbol{\mu}_+ = [1, 1]^\top$, $\boldsymbol{\Sigma}_+ = \boldsymbol{I}$
-1 class: 2-d normal distribution with $\boldsymbol{\mu}_- = [-1, -1]^\top$, $\boldsymbol{\Sigma}_- = \boldsymbol{I}$
Each class has 50 points: 10 for training, 40 for test



Classification Result with Different ε



Classification Result with Different Training Set



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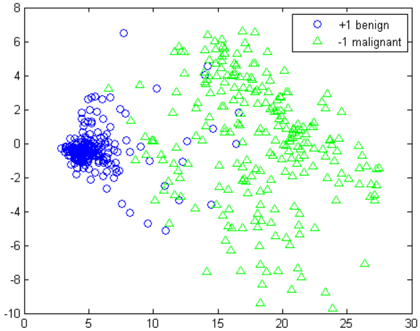
Wisconsin breast cancer data from UCI dataset:

- 444 benign(+1) class data, 239 malignant (−1) class data
- 9-dimensional features
- Use PCA to show the first 2 principle components



Wisconsin Breast Cancer Data Classification Result

Table : 20% training, 80% test

	$\varepsilon = 0.01$	$\varepsilon = 0.05$	$\varepsilon = 0.1$	$\varepsilon = 0.2$
Test Set Accuracy	96.52%	95.24%	95.24%	95.24%
SDP Running Time	35.1723	34.0854	28.6718	28.9409
SOCP Running Time	1.6846	1.6724	1.9918	2.1012

Table : 90% training, 10% test

	$\varepsilon = 0.01$	$\varepsilon = 0.05$	$\varepsilon = 0.1$	$\varepsilon = 0.2$
Test Set Accuracy	98.53%	98.53%	97.06%	97.06%
SDP Running Time	216.1927	182.9951	189.6738	164.3278
SOCP Running Time	9.3391	10.9000	12.4002	13.8963

Conclusions

- The robust chance-constrained SVM is to ensure the small probability of misclassification for the uncertain data.
  - The exact probability distribution of the random variables are unknown.
  - Some properties of the distribution are known, for example, the moments information.

- When the mean and covariance of the data points are known, the RCC-SVM can be reformulated as both SDP and SOCP models.
  - The SDP and SOCP models are equivalent, which could be proved theoretically and by experiments.
  - The SOCP model runs more efficiently than SDP model.

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Thank You!